EXISTENCE OF PARAMETRICALLY BOUND WAVEGUIDES AND SOLITONS IN THE PRESENCE OF THREE-FREQUENCY WAVE INTERACTION

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The resonance interaction of three waves whose frequencies are related by the equation $\omega_1 + \omega_2 = \omega_3$ can take place in media having quadratic nonlinearity. One of the most interesting consequences of such an interaction is the formation of specific propagation under certain conditions, when the interaction has a reactive nature for the most part. In this case compensation of the diffractional divergence (because of nonlinear variation of the phase velocities) of confined beams can occur with the formation of bound waveguides, and compensation of the dispersional spreading of short pulses can occur with the formation of bound solitons [1-3].

In a number of particular cases it has been possible to find the profiles of the waveguides and solitons: analytically (the structure of one mode of solitons in the presence of phase detuning [1]) or by numerical methods (the shape of cylindrical three-frequency waveguides [2] and of one-dimensional waveguides in the degenerate case of $\omega_1 = \omega_2$ [3] with phase synchronism, as well as one mode of cylindrical beams with detuning of the phase velocities [3]). In the general case, however, the question of the existence of waveguides and solitons has remained open.

In the present report we demonstrate the existence of a two-parameter family of waveguide and soliton solutions of the system of equations describing the three-frequency interaction of waves in a nondissipative dispersive medium:

$$\begin{aligned} &\frac{\partial v_1}{\partial z} + iD_1\Delta_\perp v_1 = -i\gamma_1 v_3 v_2^* e^{i\Delta z};\\ &\frac{\partial v_2}{\partial z} + iD_2\Delta_\perp v_2 = -i\gamma_2 v_3 v_1^* e^{i\Delta z};\\ &\frac{\partial v_3}{\partial z} + iD_3\Delta_\perp v_3 = -i\gamma_3 v_1 v_2 e^{-i\Delta z}, \end{aligned} \tag{1}$$

i.e., limited solutions of the type $v_j = y_j(x)e^{-i\Gamma_j z}$ $(\Gamma_1 + \Gamma_2 - \Gamma_3 = -\Delta)$. Here v_j are the complex amplitudes of waves propagating along the z axis; $\Delta_\perp = x^{-m}(\partial/\partial x)(x^m\partial/\partial x)$; $\Delta = k_1 + k_2 - k_3$ is the detuning of the average values of the wave vectors. For m = 0 (the plane case) the system of equations (1) describes the interaction of pulses with the condition of equality of the group velocities of the waves; $D_j = {}^1/_2(\partial^2 k_j/\partial\omega^2{}_j)$ are the coefficients of diffusion of the wave packets; $x = t - z/u$ is the associated coordinate. For m = 1 the system of equations (1) describes the interaction of axisymmetric beams, and then $D_j = 1/2k_j$ are the coefficients of transverse amplitude diffusion.

It is shown in the report that for all positive parameters $D_j^{-1}\Gamma_j$ $(\Gamma_1 + \Gamma_2 - \Gamma_3 = -\Delta)$ there exist real functions $y_j(x)$, which are positive in the semiinterval $0 \le x < \infty$ and are reduced to zero only at infinity, such that

$$\dot{y}_j(0) = y_j(\infty) = 0 \tag{2}$$

and $v_j = y_j(x)e^{-i\Gamma_j z}$ are solutions of the system (1). In this case the functions $y_j(x)$ satisfy the system of equations

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$$\begin{aligned}
\Delta_{\perp} y_1 - \Gamma_1 D_1^{-1} y_1 &= -\gamma_1 D_1^{-1} y_3 y_2; \\
\Delta_{\perp} y_2 - \Gamma_2 D_2^{-1} y_2 &= -\gamma_2 D_2^{-1} y_3 y_1; \\
\Delta_{\perp} y_3 - \Gamma_3 D_3^{-1} y_3 &= -\gamma_3 D_2^{-1} y_1 y_2.
\end{aligned} \tag{3}$$

This means that for any phase detuning Δ with positive coefficients of diffusion D_j there exists a set of fundamental modes of waveguides and solitons differing from one another by the magnitude of the phase velocity and the shape of the amplitude profile. The coefficients of diffusion D_j of the wave packets can have different signs. Conditions for the formation of solitons are also possible if they are all negative. In this case, in contrast to the case of positive D_j, the solitons will be accelerated.

We note that the equations (1) allow the simultaneous sign change of two of the complex amplitudes v_j. Therefore, when $D_j > 0$, besides the solutions with positive amplitude profiles $y_j(x)$, which can conditionally be called cophase solutions, there exist solutions when the amplitude profiles of two of the waves are negative (in antiphase to the third wave). Similarly, when $D_j < 0$, besides the negative solutions $y_j(x)$ the system (3) has solutions when two of the functions are positive. Such solutions can have different stabilities relative to small changes in the initial conditions.

1. Existence of Solutions for m = 0

The main idea of the proof consists in drawing on the variational principle, and it has been used earlier to prove the existence of a solution of one second-order differential equation [4].

Without a restriction of generality we will take all the coefficients in the system (3) as equal to unity. Let us consider the problem at the minimum of the functional

$$J(y) = \int_0^\infty \sum_{j=1}^3 \left(y_j^2 + \dot{y}_j^2\right) dx = \min, \tag{4}$$

where $y_j(x)$ belong to a class K of functions which are nonnegative, continuous, and have piecewise-continuous derivatives in $0 \le x < \infty$ satisfying the conditions (2), the normalization condition

$$\int_0^\infty y_1 y_2 y_3 dx = 1, \tag{5}$$

and are such that the integral (4) exists.

The existence of the integral (5) follows from the existence of the integral (4) and the elementary inequality

$$y_j^2(x) = -2\int_x^\infty y_j \dot{y}_j d\xi \leqslant \int_0^\infty (y_j^2 + \dot{y}_j^2)\, d\xi. \tag{6}$$

For functions $y_j \in K$ it follows from (5) and (6) that $J(y) \ge 1$. Then there exists an exact lower bound $\lambda = \inf\limits_{y \in K} J(y) \geqslant 1$ and a sequence of functions $\{y_j^{(n)}\} \in K$ for which $\lim\limits_{n \to \infty} J(y^{(n)}) = \lambda$. Let $J(y^{(n)}) \le c^2$. From (6) we get

$$y_j^{(n)^2}(x) \leqslant c^2, \; n = 1,\; 2,\; \ldots. \tag{7}$$

On the other hand, for any x_1 and x_2

$$|y_j^{(n)}(x_2) - y_j^{(n)}(x_1)|^2 = \left(\int_{x_1}^{x_2} \dot{y}_j^{(n)} dx\right)^2 \leqslant J(y^{(n)})(x_2 - x_1) \leqslant c^2 (x_2 - x_1).$$

According to the Arzelà theorem, from the sequence $\{y_j^{(n)}\}$ one can choose a sequence which uniformly converges in any finite interval to the continuous functions $y_j(x)$.

The problem now consists in proving that with the choice of suitable positive constants β_j the functions $\beta_j y_j(x)$ will be a solution of the problem (2), (3). For this we consider the system of equations

$$\begin{aligned} &\ddot{u}_1^{(n)} - u_1^{(n)} + \alpha_1^{(n)} y_3^{(n)} y_2^{(n)} = 0; \\ &\ddot{u}_2^{(n)} - u_2^{(n)} + \alpha_2^{(n)} y_3^{(n)} u_1^{(n)} = 0; \\ &\ddot{u}_3^{(n)} - u_3^{(n)} + \alpha_3^{(n)} u_1^{(n)} u_2^{(n)} = 0 \end{aligned} \tag{8}$$

with the boundary conditions

$$\dot{u}_j(0) = u_j(\infty) = 0. \tag{9}$$

The solution of the problem (8), (9) has the form

$$\begin{aligned} u_1^{(n)} &= \alpha_1^{(n)} \int_0^\infty G(x, \xi)\, y_3^{(n)}(\xi)\, y_2^{(n)}(\xi)\, d\xi; \\ u_2^{(n)} &= \alpha_2^{(n)} \int_0^\infty G(x, \xi)\, y_3^{(n)}(\xi)\, u_1^{(n)}(\xi)\, d\xi; \\ u_3^{(n)} &= \alpha_3^{(n)} \int_0^\infty G(x, \xi)\, u_1^{(n)}(\xi)\, u_2^{(n)}(\xi)\, d\xi, \end{aligned} \tag{10}$$

where $G(x, \xi) = (1/2)(e^{-|x-\xi|} + e^{-(x+\xi)})$ is the Green function of the operator $L(u) = \ddot{u} - u$ for the boundary conditions (9).

Using the representation (10), one can prove that the integral $J(u^{(n)})$ exists. We choose the constants $\alpha_j^{(n)}$ from the conditions

$$\int_0^\infty u_1^{(n)} y_2^{(n)} y_3^{(n)} dx = \int_0^\infty u_1^{(n)} u_2^{(n)} y_3^{(n)} dx = \int_0^\infty u_1^{(n)} u_2^{(n)} u_3^{(n)} dx = 1$$

and we prove that the solution of the problem (8), (9) decreases the functional (4), i.e.,

$$J(u^{(n)}) \leqslant J(y^{(n)}).$$

After multiplication of Eqs. (8) by $u_j^{(n)}$, summation, and integration, we obtain

$$J(u^{(n)}) = \sum_{j=1}^{3} \alpha_j^{(n)}. \tag{11}$$

On the other hand, multiplication of (8) by $y_j^{(n)}$ and integration gives

$$J(u^{(n)}) + J(y^{(n)}) - J(u^{(n)} - y^{(n)}) = 2 \sum_{j=1}^{3} \alpha_j^{(n)}. \tag{12}$$

The following relations result directly from (11), (12), and (6):

$$\begin{aligned} &J(u^{(n)}) \leqslant J(y^{(n)}), \quad u_j^{(n)} \in K; \\ &\lim_{n\to\infty} J(u^{(n)}) = \lim_{n\to\infty} J(y^{(n)}) = \lim_{n\to\infty} \sum_{j=1}^{3} \alpha_j^{(n)} = \lambda; \\ &\lim_{n\to\infty} J(u^{(n)} - y^{(n)}) = 0; \\ &\lim_{n\to\infty} |u_j^{(n)}(x) - y_j^{(n)}(x)| = 0, \qquad x \in [0, \infty). \end{aligned} \tag{13}$$

If one uses the uniform convergence of the sequence $\{y_j^{(n)}\}$ to continuous functions in any finite interval and one uses the representation (10), then one can test the existence of the limits $\lim\limits_{n\to\infty} \alpha_j^{(n)} = \alpha_j$.

Let us return to the determination of the functions $u_j^{(n)}$ of (10). We divide the interval of integration into [0, X] and [X, ∞], and then for $x < X$ we have

$$\left| y_1^{(n)}(x) - \alpha_1^{(n)} \int_0^X G(x,\xi)\, y_3^{(n)}(\xi)\, y_2^{(n)}(\xi)\, d\xi \right| \leqslant |y_1^{(n)} - u_1^{(n)}| + \alpha_1^{(n)} \int_X^\infty G(x,\xi)\, y_3^{(n)}(\xi)\, y_2^{(n)}(\xi)\, d\xi.$$

From the uniform convergence of the sequence $\{y_j^{(n)}\}$ in the segment [0, X], the last equation in (13), and (7) it follows that

$$\left| y_1(x) - \alpha_1 \int_0^X G(x,\xi)\, y_3(\xi)\, y_2(\xi)\, d\xi \right| \leqslant \alpha_1 c^2 \operatorname{ch} x e^{-X}.$$

Taking X to infinity, we obtain

$$y_1(x) = \alpha_1 \int_0^\infty G(x,\xi)\, y_3(\xi)\, y_2(\xi)\, d\xi. \tag{14}$$

In a similar way we establish the equations

$$y_2(x) = \alpha_2 \int_0^\infty G(x,\xi)\, y_3(\xi)\, y_1(\xi)\, d\xi; \tag{15}$$

$$y_3(x) = \alpha_3 \int_0^\infty G(x,\xi)\, y_1(\xi)\, y_2(\xi)\, d\xi. \tag{16}$$

By its construction $y_j^{(n)}(x) > 0$ in $[0, \infty)$ and $\alpha_j > 0$ and $y_j(x) \geq 0$ in $[0, \infty)$. The limiting functions satisfy the conditions (2) and by virtue of (5) they cannot be identically equal to zero. The right sides in (14)-(16) can be differentiated twice with respect to x, as a result of which the functions $\tilde{y}_j = (\sqrt{\alpha_1\alpha_2\alpha_3/\alpha_j})\, y_j$ will satisfy Eqs. (3) with unit coefficients and the boundary conditions (2).

2. Existence of Solutions for m = 1

By analogy with Sec. 1 we set the problem at the minimum of the functional

$$J(y) = \int_0^\infty \sum_{j=1}^3 (y_j^2 + \dot{y}_j^2)\, x\,dx = \min$$

with the normalization condition

$$\int_0^\infty y_1 y_2 y_3 x\,dx = 1$$

in the same class of functions as for the plane case. But now the minimizing sequence $\{y_j^{(n)}\}$ can also converge nonuniformly in any finite interval. However, one can always construct a minimizing sequence $\{u_j^{(n)}\}$ such that it converges uniformly in any segment [0, X]. To prove this it is enough to prove that the integral

$$J(u^{(n)}) = \int_0^\infty \sum_{j=1}^3 (u_j^{(n)2} + \dot{u}_j^{(n)2})\, dx \tag{17}$$

is uniformly bounded. As the functions $u_j^{(n)}$ we take the solution of the system

$$\begin{aligned} &\ddot{u}_1^{(n)} + \frac{1}{x}\dot{u}_1^{(n)} - u_1^{(n)} + \alpha_1^{(n)} y_3^{(n)} y_2^{(n)} = 0; \\ &\ddot{u}_2^{(n)} + \frac{1}{x}\dot{u}_2^{(n)} - u_2^{(n)} + \alpha_2^{(n)} y_3^{(n)} u_1^{(n)} = 0; \\ &\ddot{u}_3^{(n)} + \frac{1}{x}\dot{u}_3^{(n)} - u_3^{(n)} + \alpha_3^{(n)} u_1^{(n)} u_2^{(n)} = 0 \end{aligned} \tag{18}$$

with the boundary conditions (9). Then the representation (10) is valid, where the Green function of the operator $L(u) = \ddot{u} + (1/x)\dot{u} - u$, $\dot{u}(0) = u(\infty) = 0$ has the form

$$G(x, \xi) = \begin{cases} \xi I_0(\xi) K_0(x), & 0 \leqslant \xi \leqslant x < \infty, \\ \xi K_0(\xi) I_0(x), & 0 \leqslant x \leqslant \xi < \infty. \end{cases}$$

It turns out that $y_j^{(n)} \in L_2[0, \infty)$. In fact,

$$\int_0^\infty y_j^{(n)2} dx = -\int_0^\infty (\dot{y}_j^{(n)2})\, x dx = -2\int_0^\infty y_j^{(n)} \dot{y}_j^{(n)} x dx \leqslant c^2. \tag{19}$$

Using the inequality (19) and the representation of the solution of the system (18) in the form (10), we can make sure of the uniform boundedness of the integral (17). Then the uniformly converging sequence $\{u_j^{(n)}\}$ can be taken as the initial minimizing sequence. The further course of the proof coincides with that carried out for the plane case.

LITERATURE CITED

1. Yu. N. Karamzin and A. P. Sukhorukov, "On the mutual focusing of strong light beams in media with quadratic nonlinearity," Zh. Éksp. Teor. Fiz., 68, No. 3, 835-847 (1975).
2. Yu. N. Karamzin and A. P. Sukhorukov, "On the self-capture of light beams in parametrically bound waveguides," Zh. Tekh. Fiz., Pis'ma Red., 1, No. 16, 737-741 (1975).
3. Yu. N. Karamzin and A. P. Sukhorukov, "Nonlinear interaction of diffracting light beams in a medium with quadratic nonlinearity," Zh. Éksp. Teor. Fiz., Pis'ma Red., 20, No. 11, 734-739 (1974).
4. Z. Nehary, "Characteristic values associated with a class of nonlinear second-order differential equations," Acta Math., 10S, 141 (1961).